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Inverse Problems and Sound Reproduction

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Summary

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- A tutorial introduction to solutions of the wave equation in spherical coordinates
- Introduction to spherical harmonic decomposition
- Sound field reproduction using discrete sources and the associated linear algebra
- Ill conditioning of inverse problems
- Sound field recording
- Connections with the “active control” of sound
- Progression to the analysis of continuous source/sensor arrangements

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The Wave Equation and its Solution in Spherical Coordinates

The Acoustic Wave Equation

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- From considerations of the conservation of momentum and mass…

$$\nabla^2 p - \frac{1}{c_0}\frac{\partial^2 p}{\partial t^2} = 0$$

is the acoustic wave equation that describes the pressure field in a space in any arbitrary coordinate system

- For a harmonic time dependence, where the wave equation reduces to the Helmholtz equation which can be written in terms of the complex pressure as

$$\boxed{\nabla^2 p + k^2 p = 0}$$

$k = \omega / c_0 = 2\pi / \lambda$ is the free-field acoustic wavenumber

The Spherical Coordinate System

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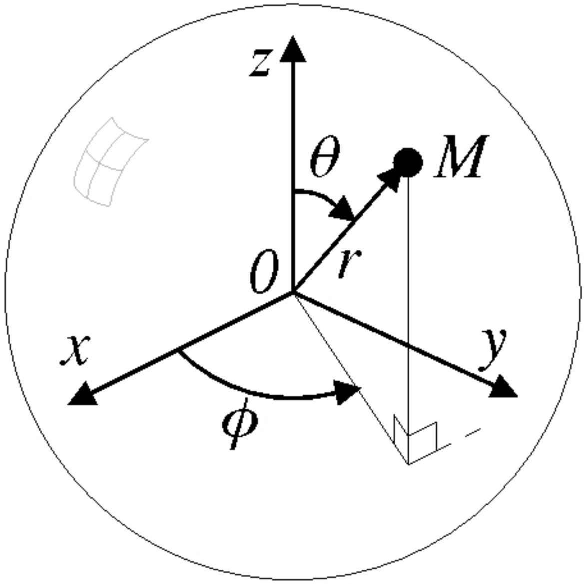
- Geometrical relationships between Cartesian (x, y, z) coordinates and spherical (r, θ, ϕ) coordinates…

$$x = r\sin(\theta)\cos(\phi)$$
$$y = r\sin(\theta)\sin(\phi)$$
$$z = r\cos(\theta)$$



- So the Helmholtz equation becomes…

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right)+\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial p}{\partial\theta}\right)+\frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 p}{\partial\phi^2}+k^2p=0$$

Spherical Coordinates: Special Case

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- For spherically symmetric sound fields (e.g. from a point monopole)

$$\frac{\partial p}{\partial \theta} = \frac{\partial^2 p}{\partial \phi^2} = 0$$

so that the Helmholtz equation becomes…

$$\frac{2}{r}\frac{\partial^2 p}{\partial r^2} + k^2 p = 0$$

with solutions of the form…

$$p = A\frac{e^{ikr}}{r} + B\frac{e^{-ikr}}{r}$$

A and B are the amplitudes of outward and inward travelling waves respectively

General Spherical Solution

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- Use the "method of separation of variables" to change the partial differential equation into a number of independent ordinary differential equations

- It is assumed that

$$p(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

- Where $p(r,\theta,\phi)$ is the complex pressure and harmonic time dependence of the form $e^{-i\omega t}$ has been assumed

- See for example "Fourier Acoustics" (E.G. Williams) Ch.6

General Spherical Solution

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- Separating variables yields three ordinary differential equations giving dependence of pressure on r, θ, ϕ …….

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$$

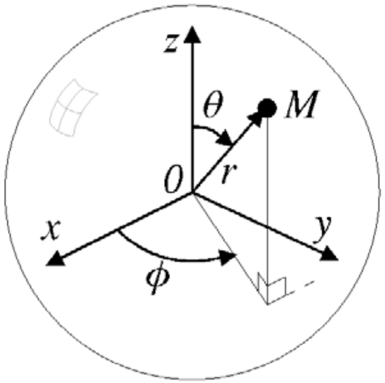
$$\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2(\theta)}\right]\Theta = 0$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + k^2R - \frac{n(n+1)}{r^2}R = 0$$

Solution in terms of Spherical Harmonics and Spherical Hankel Functions

General Spherical Solution: ϕ dependence

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0$$



- The solution for ϕ dependence is…

$$\boxed{\Phi(\phi) = \Phi_1 e^{im\phi} + \Phi_2 e^{-im\phi}}$$

note: $\Phi(\phi)$ is continuous and periodic around the sphere for integer m

General Spherical Solution: θ dependence

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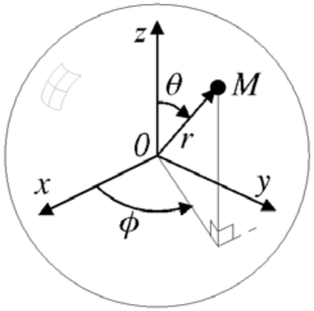
- The solution for θ dependence involves a transformation of variables, such that if $\eta = \cos(\theta)$ then the equation for Θ becomes…

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d\Theta}{d\eta}\right] + \left[n(n+1) - \frac{m^2}{1-\eta^2}\right]\Theta = 0$$

which has solutions of the form



$$\boxed{\Theta(\theta) = \Theta_1 P_n^m(\cos(\theta))}$$

where $P_n^m(.)$ are Legendre functions of the 1st kind of order n and mode m

The Legendre functions

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- These are analytical expressions, depending on the integers n and m, given by functions such as:

$$P_0^0 = 1$$

$$P_1^0 = \cos\theta$$

$$P_2^0 = \frac{1+3\cos 2\theta}{4}$$

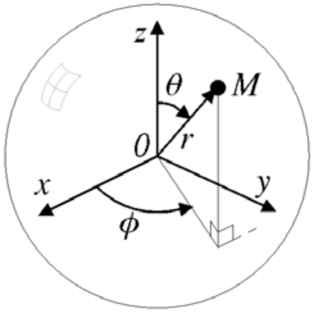
$$P_3^0 = \frac{-3\cos\theta + 5\cos^3\theta}{2}$$

$$P_1^1 = -\sin\theta$$

$$P_2^1 = -3\cos\theta\sin\theta$$

$$P_3^1 = \frac{3(1-5\cos^2\theta)\sin\theta}{2}$$

$$P_4^1 = \frac{5(3\cos\theta - 7\cos^3\theta)\sin\theta}{2}$$

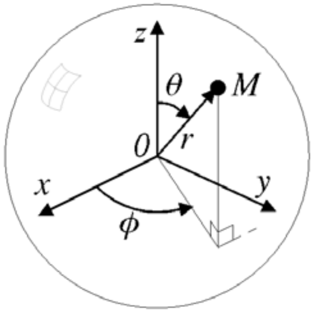


General Spherical Solution: the angular function

- The dependence on θ and ϕ can be expressed in terms of spherical harmonics…

$$Y_n^m(\theta,\phi)=\sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}}P_n^m(\cos(\theta))e^{im\phi}$$

- Thus the angular dependence of the sound field can be expressed as a combination (weighted sum) of spherical harmonics



The Spherical Harmonics

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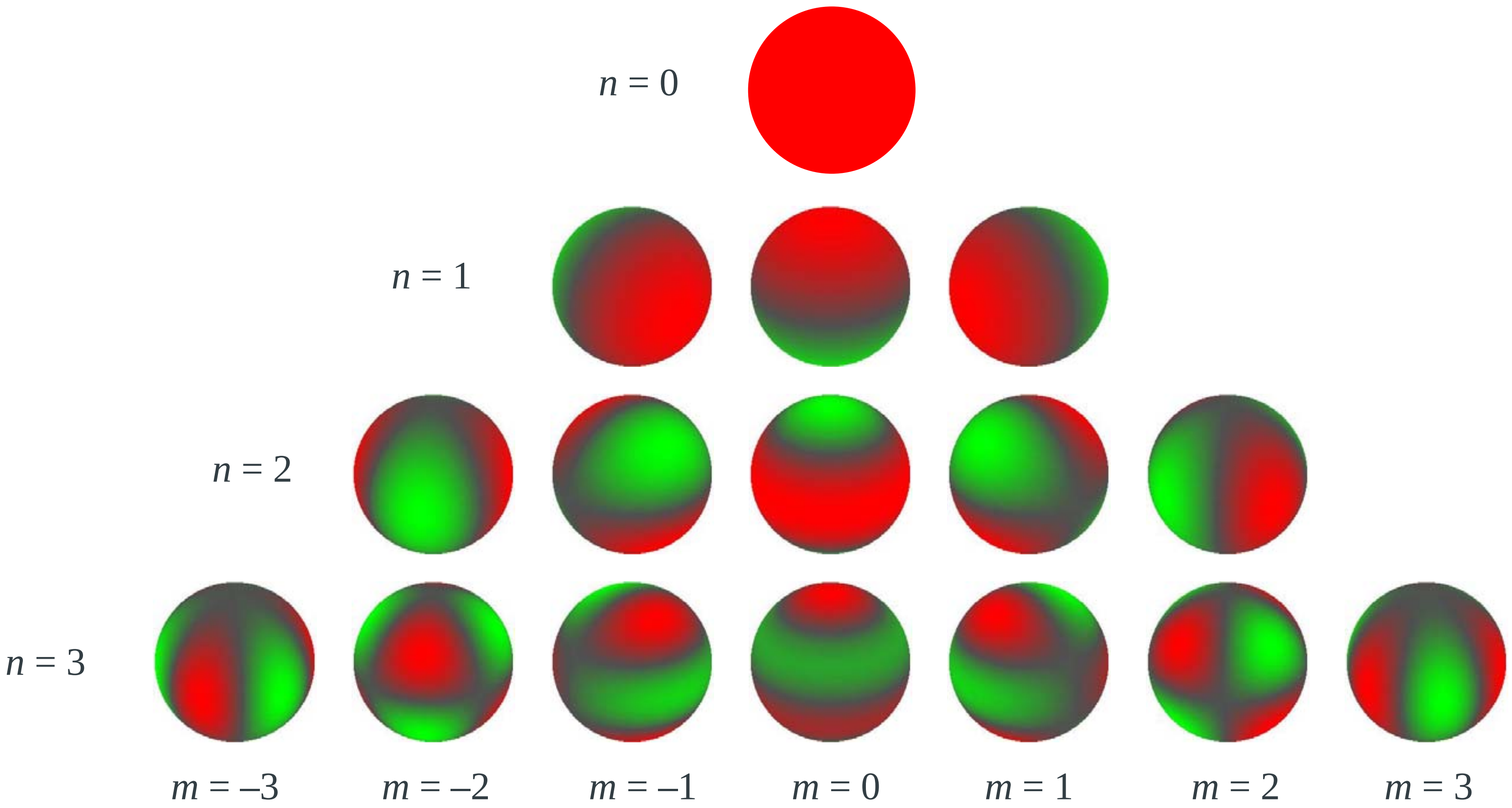
- The angular dependence of a sound field can be expressed as a series of spherical Harmonics which take the form…



The Spherical Harmonics

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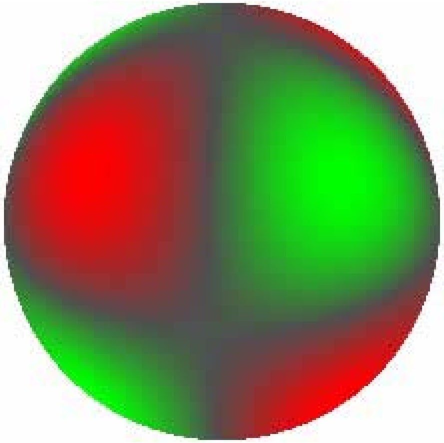
- The angular dependence of a sound field can be expressed as a series of spherical Harmonics which take the form……
- and with harmonic time dependence…

$m = 2$
$n = 3$



General Spherical Solution: the radial function

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- The solution for radial dependence can be written as a combination of spherical Hankel functions of the 1st and 2nd kind…

$$h_n^{(1)} = j_n(kr) + iy_n(kr)$$
$$h_n^{(2)} = j_n(kr) - iy_n(kr)$$

where j_n and y_n are spherical Bessel functions, given by

$$j_n(kr) = \sqrt{\frac{\pi}{2kr}} J_{n+0.5}(kr) \text{ and } y_n(kr) = \sqrt{\frac{\pi}{2kr}} Y_{n+0.5}(kr)$$

where J_n and Y_n are Bessel functions.

The Spherical Hankel Function

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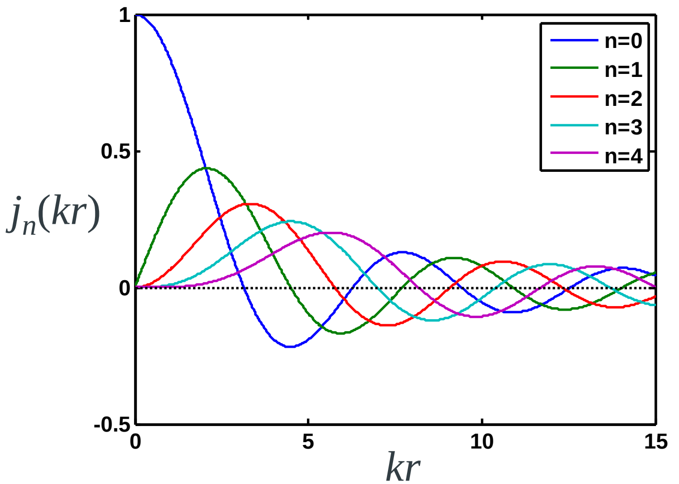
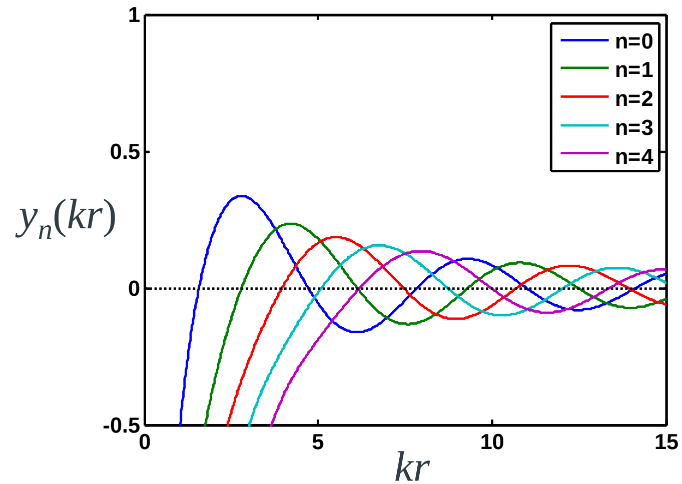
- The radial dependence of a sound field can be expressed as a series of spherical Hankel functions
- The real and imaginary parts of the first 5 orders of spherical Hankel functions have the form…





- Note that the spherical Hankel function of the 2nd kind is simply the complex conjugate of the 1st kind, and that h_0 represents the spherically symmetric wave field

General Spherical Solution

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- Combining the angular and radial functions, any solution of the wave equation, and hence any sound field, can be written…

$$p(r,\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(C_{mn} h_n^{(1)}(kr) + D_{mn} h_n^{(2)}(kr) \right) Y_n^m(\theta,\phi)$$

(harmonic time dependence assumed)

- C_{mn} and D_{mn} are the complex amplitudes of outgoing and incoming waves respectively with angular order n and mode m.

Fourier Bessel Expansion

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- An alternative form of the solution, using the definitions of the Hankel functions

$$p(r,\theta,\phi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[(C_{mn}+D_{mn})j_n(kr)+i(C_{mn}-D_{mn})y_n(kr)\right]Y_n^m(\theta,\phi)$$

$$=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left[B_{mn}j_n(kr)+iA_{mn}y_n(kr)\right]Y_n^m(\theta,\phi)$$

- But since $y_n(kr)$ is infinite at $r=0$ and if the region is source free

$$p(r,\theta,\phi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}B_{mn}j_n(kr)Y_n^m(\theta,\phi)$$

The Spherical Harmonics

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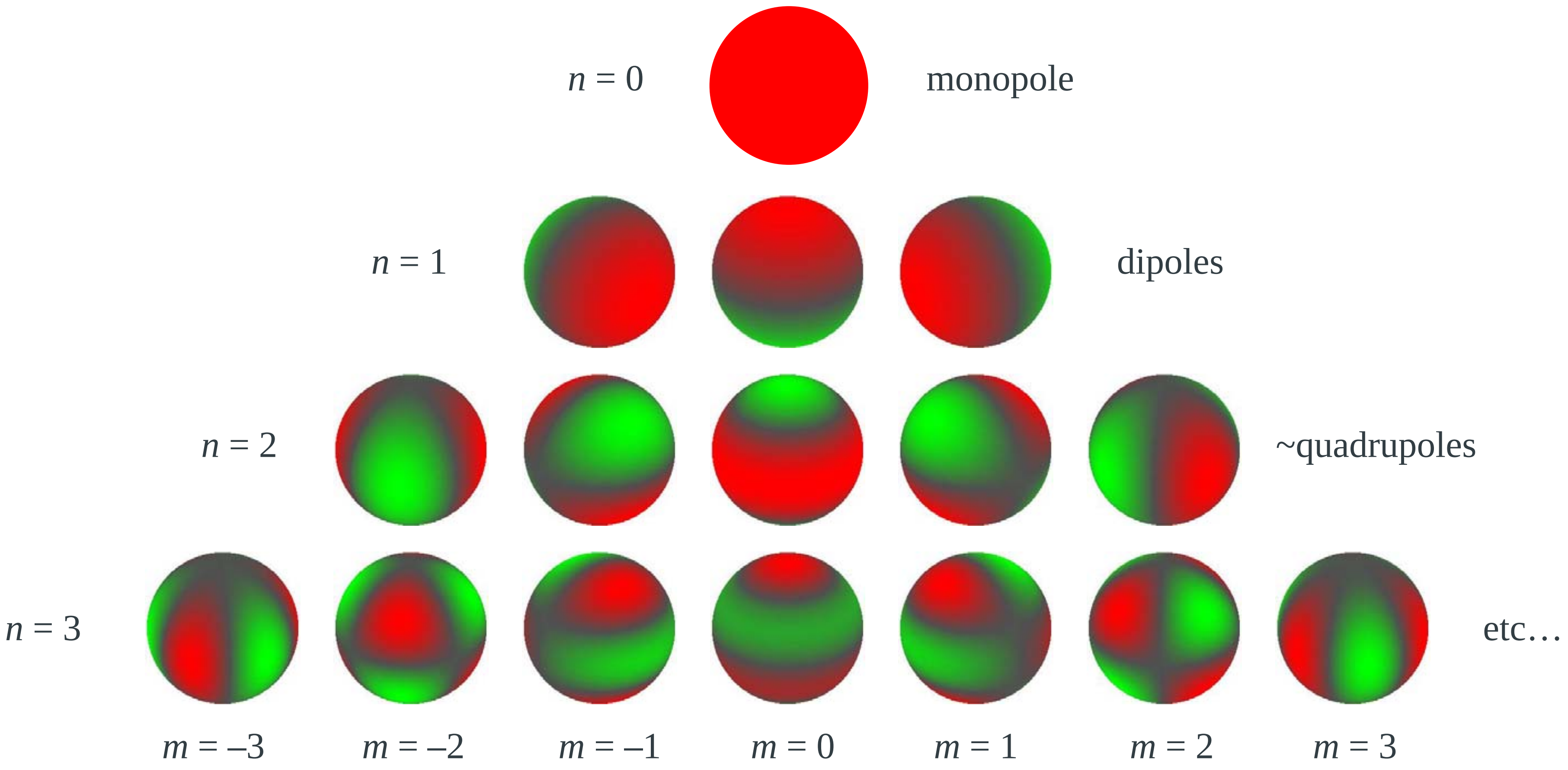
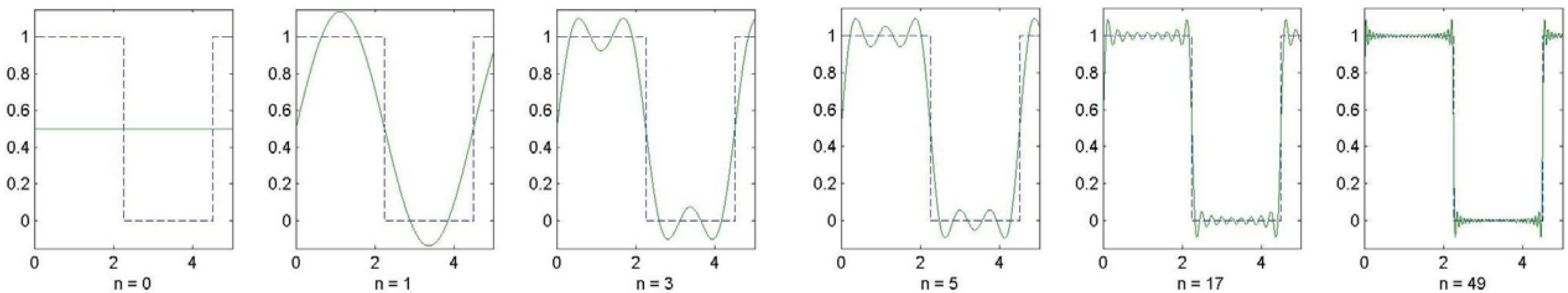
- Each order of spherical harmonic may be associated with the outgoing wave generated by elemental source types…

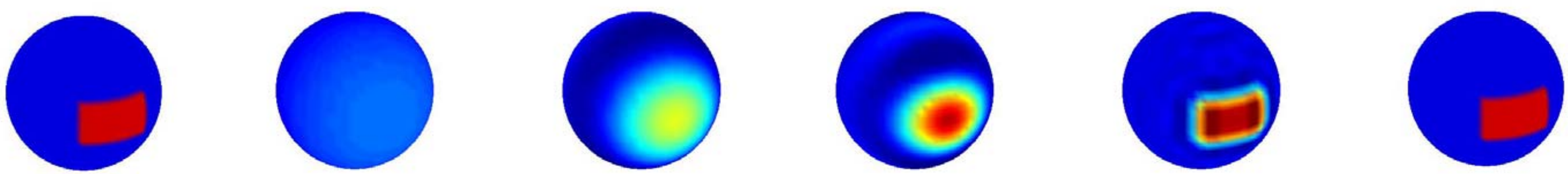


Spherical Harmonics and Fourier Series

- Fourier series



- Spherical harmonic expansion (generalised Fourier series)



Target n=1 n=3 n=5 n=17 n=49

Orthogonality of spherical harmonics

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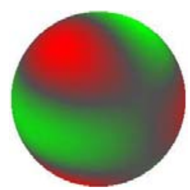
$$\int_\Omega Y_n^m(r,\theta,\phi) Y_p^q(r,\theta,\phi)^* d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} Y_n^m(r,\theta,\phi) Y_p^q(r,\theta,\phi)^* \sin\theta d\theta$$

$$= \delta_{np}\delta_{mq}$$

where δ_{np} is the Kronecker delta function $\delta_{np} = \begin{cases} 1 & n = p \\ 0 & n \neq p \end{cases}$

$\int_\Omega$   $^* d\Omega = \delta_{np}\delta_{mq}$

Representation of a plane wave in terms of spherical harmonics

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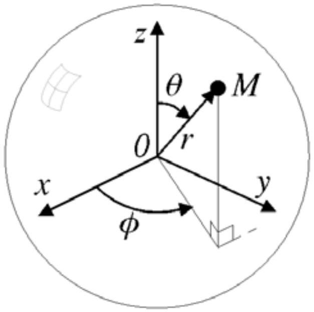
- The Jacobi-Anger expansion shows that a plane wave can be expressed in terms of a series of spherical harmonics of infinite order

$$e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}} = 4\pi \sum_{n=0}^{\infty} i^n j_n(kr) \sum_{m=-n}^{n} Y_n^m(\theta_r, \phi_r) Y_n^m(\theta_k, \phi_k)^*$$

….but an increasingly good representation can be produced as the order of a finite expansion is increased.



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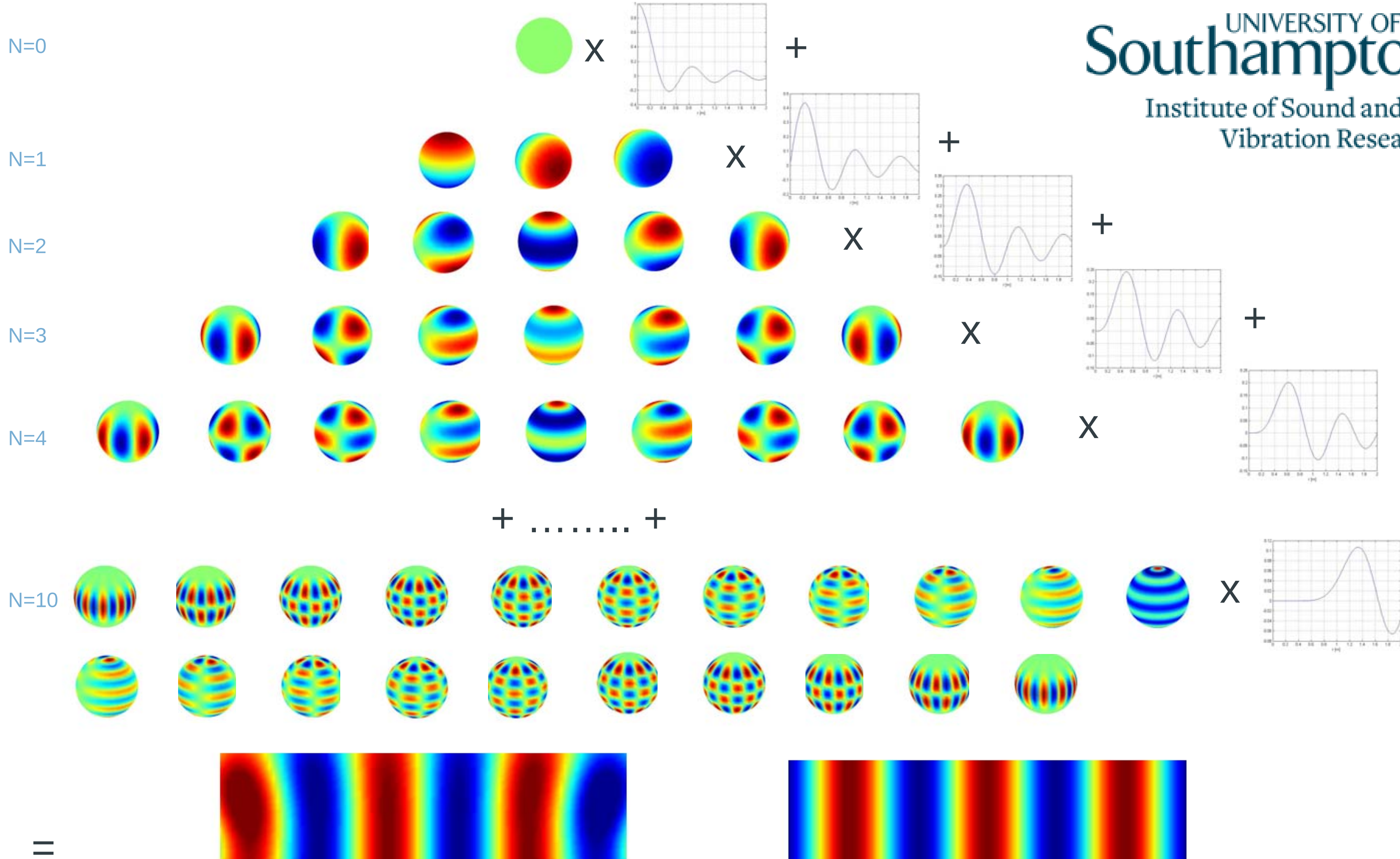
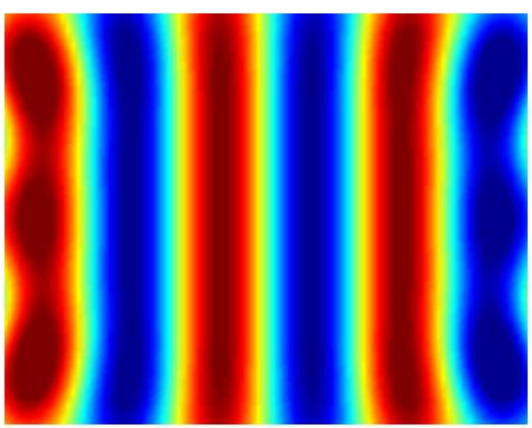
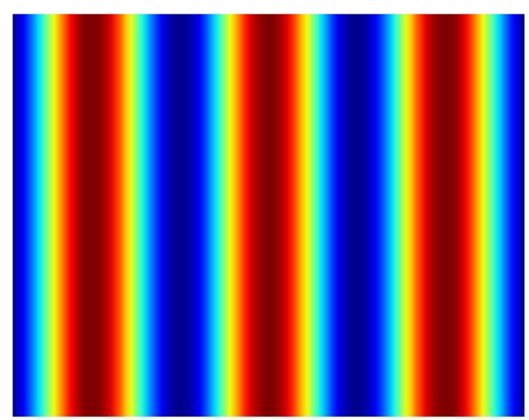



Reproduced field


Target field

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Sound Reproduction using Spherical Harmonics

Some acknowledgements

- There has been a number of approaches including “ambisonics”, “wavefield synthesis”, “vector based amplitude panning” and “least squares methods”.
- See, for example, Gerzon 1973, Berkhout 1988, Kirkeby and Nelson 1993, de Vries et al 1994, Bamford and Vanderkooy 1995, Pulkki 1997, Ise 1999, Farina et al 2001, Ward and Abhayapala 2001, Daniel et al 2003 et al, Spors et al 2004, Betlehem et al 2005, Poletti 2005, Gauthier and Berry 2006, Ahrens et al 2008, Cho et al 2008, Fazi 2010 and many more.
- In what follows, we will concentrate on making use of the spherical harmonic expansion.

Array of loudspeakers

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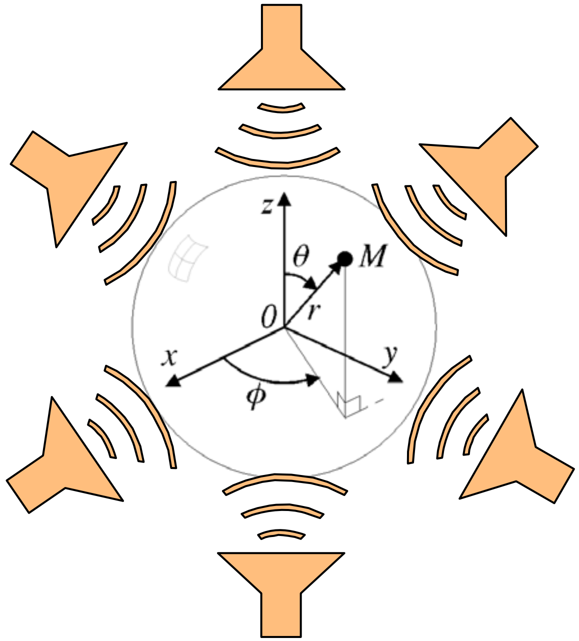

- The reproduced field is the linear superposition of the fields generated by the L loudspeakers.
- Each loudspeaker is assumed to generate a plane wave arriving from the direction $\vec{\mathbf{k}}_\ell = [k, \theta_\ell, \phi_\ell]$

$$\hat{p}(r,\theta,\phi) = \sum_{\ell=1}^{L} w_\ell \, e^{-i\vec{\mathbf{k}}_\ell \cdot \vec{\mathbf{r}}}$$

$$e^{-i\vec{\mathbf{k}}_\ell \cdot \vec{\mathbf{r}}} = 4\pi \sum_{n=0}^{\infty} i^{-n} j_n(kr) \sum_{m=-n}^{n} Y_n^m(\theta_r,\phi_r) Y_n^m(\theta_\ell,\phi_\ell)^*$$

$$\hat{p}(r,\theta,\phi) = 4\pi \sum_{n=0}^{\infty} i^{-n} j_n(kr) \sum_{m=-n}^{n} Y_n^m(\theta_r,\phi_r) \sum_{\ell=1}^{L} w_\ell \, Y_n^m(\theta_\ell,\phi_\ell)^*$$

Coefficients for the reproduced field

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- The Jacobi-Anger expansion and the general expression:

$$\hat{p}(r,\theta,\phi) = 4\pi \sum_{n=0}^{\infty} i^{-n} j_n(kr) \sum_{m=-n}^{n} Y_n^m(\theta_r,\phi_r) \sum_{\ell=1}^{L} w_\ell \, Y_n^m(\theta_\ell,\phi_\ell)^*$$

$$\hat{p}(r,\theta,\phi) = \sum_{n=0}^{\infty} j_n(kr) \sum_{m=-n}^{n} Y_n^m(\theta_r,\phi_r) \hat{B}_{mn}$$

Multiply both equations by $Y_n^m(\theta_r,\phi_r)^*$, integrate over a unit sphere and apply the orthogonality property of the spherical harmonics to give

$$\hat{B}_{mn} = 4\pi i^{-n} \sum_{\ell=1}^{L} w_\ell \, Y_n^m(\theta_\ell,\phi_\ell)^*$$

$$n = 1,2,\ldots\infty \quad |m| \leq n$$

One equation for each coefficient,

but we choose a finite number $(N+1)^2$ of coefficients

Matrix formulation

$$\hat{B}_{mn} = 4\pi i^{-n} \sum_{\ell=1}^{L} w_\ell \, Y_n^m(\theta_\ell, \phi_\ell)^* \quad \Rightarrow \quad \hat{\mathbf{b}} = \mathbf{RYw}$$

$$\hat{\mathbf{b}} = \begin{bmatrix} \hat{B}_{11} \\ \vdots \\ \hat{B}_{NN} \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} i^0 & & 0 \\ & \ddots & \\ 0 & & i^{-N} \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} Y_0^0(\theta_1, \phi_1) & \cdots & Y_0^0(\theta_L, \phi_L) \\ \vdots & \ddots & \vdots \\ Y_N^N(\theta_1, \phi_1) & \cdots & Y_N^N(\theta_L, \phi_L) \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_L \end{bmatrix}$$

Solution ("decoding")

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Target field $$p_N(r,\theta,\phi) = \sum_{n=0}^{N} j_n(kr) \sum_{m=-n}^{n} Y_n^m(\theta_r,\phi_r) B_{mn} \;\rightarrow\; \mathbf{b}$$

Reproduced field $$\hat{p}_N(r,\theta,\phi) = \sum_{n=0}^{N} j_n(kr) \sum_{m=-n}^{n} Y_n^m(\theta_r,\phi_r) \hat{B}_{mn} \;\rightarrow\; \hat{\mathbf{b}} = \mathbf{RYw}$$

$$J = \left\| \hat{\mathbf{b}} - \mathbf{b} \right\|^2 = \left\| \mathbf{RYw} - \mathbf{b} \right\|^2$$

Cost function to be minimized

(standard Least Squares problem)

$$\mathbf{w}_0 = \mathbf{Y}^{+}\mathbf{R}^{-1}\mathbf{b}$$

$$\mathbf{Y}^{+} = (\mathbf{Y}^{\mathbf{H}}\mathbf{Y})^{-1}\mathbf{Y}^{\mathbf{H}}$$ $L<(N+1)^2$

$$\mathbf{Y}^{+} = \mathbf{Y}^{-1}$$ $L=(N+1)^2$

$$\mathbf{Y}^{+} = \mathbf{Y}^{\mathbf{H}}(\mathbf{Y}\mathbf{Y}^{\mathbf{H}})^{-1}$$ $L>(N+1)^2$

Singular Value Decomposition

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- Any matrix $\mathbf{H}$ can be expressed as the product of three matrices

$$\mathbf{H} = \mathbf{U}\Sigma\mathbf{V}^{\mathbf{H}}$$

- $\mathbf{U}$ and $\mathbf{V}$ are unitary matrices $(\mathbf{U}^{\mathbf{H}}\mathbf{U}) = \mathbf{I}$ $(\mathbf{V}^{\mathbf{H}}\mathbf{V}) = \mathbf{I}$
- Σ is a diagonal matrix, whose elements σ_n are the singular values of $\mathbf{H}$
- The condition number is the ratio between the largest and the smallest singular value. If the condition number is large, the matrix $\mathbf{H}$ is said to be *ill-conditioned*

$$\kappa(\mathbf{H}) = \sigma_N / \sigma_1$$

- In the case under consideration, if the loudspeaker arrangement is fairly uniform, then the condition number is close to unity (the matrix $\mathbf{Y}$ is well-conditioned)

Uniform sampling

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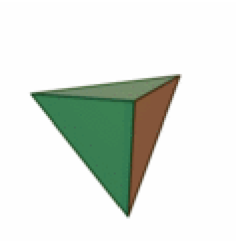
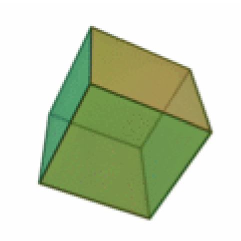
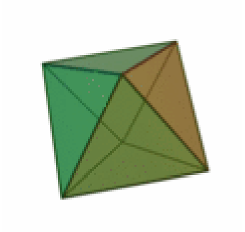
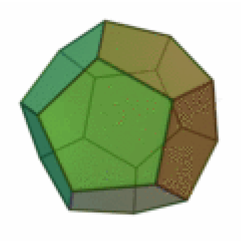
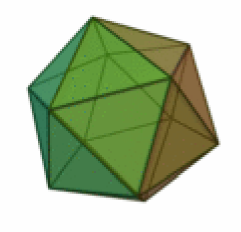
- In the special case of uniform sampling *and* when $L=(N+1)^2$ the matrix Y is *unitary* and it follows that

$$\mathbf{Y}^{\mathbf{H}}\mathbf{Y} = \mathbf{I}\left(4\pi / L\right) \qquad \mathbf{Y}^{-1} = \mathbf{Y}^{\mathbf{H}}\left(4\pi / L\right)$$

- Platonic solids: (tetrahedron, hexahedron, octahaedron, dodecahedron, icosahedron, with 4, 6, 8,12, 20 faces and 4, 8, 6, 20,12 vertices). Only the tetrahedron allows N to be chosen so that $L=(N+1)^2$

- However, note that $\mathbf{Y}^{\mathbf{H}}\mathbf{Y} \rightarrow \mathbf{I}\left(4\pi / L\right)$ as $L \rightarrow \infty$

Reconstruction of the spherical harmonics

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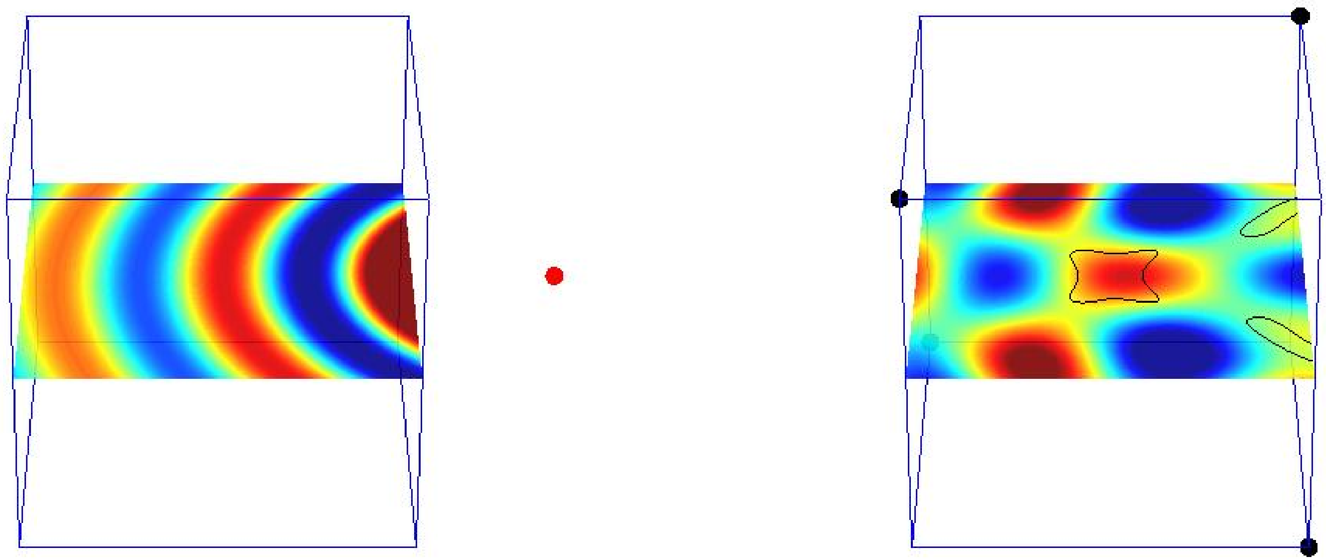
- Illustration with “Classical Ambisonics”; assume that the first four terms in the series can be measured and four loudspeakers in a 2m cube tetrahedral array are used to reproduce these terms at 200Hz (see M.A. Gerzon 1973, J. Audio Eng. Soc., 21,1-9)

Target field

Reproduced field



Reconstruction of the spherical harmonics

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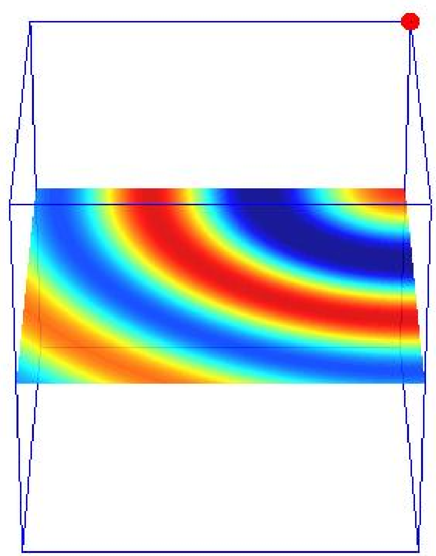
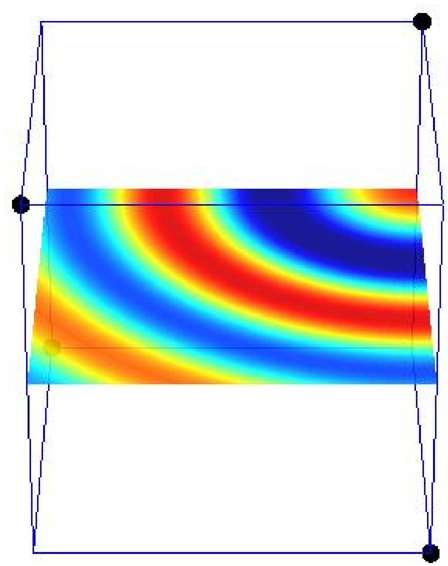


Reproduced field



A simple rule of thumb

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- The (*minimum*?) volume over which a field can be reproduced by replicating the first N spherical harmonics is given approximately by

$$r \approx N/k = Nc/2\pi f \approx 55N/f$$

- See Ward and Abhayapala 2001, Trans. IEEE 9, 697-707
- E.g. for r=1m, at 200Hz need N=4, but at 2000Hz need N= 40
- Number of loudspeakers for reproduction ~$(N+1)^2$; perhaps not a sensible strategy at high frequencies?

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Sound Field Recording

Measurement of the field and estimation of the coefficients (“Encoding”)

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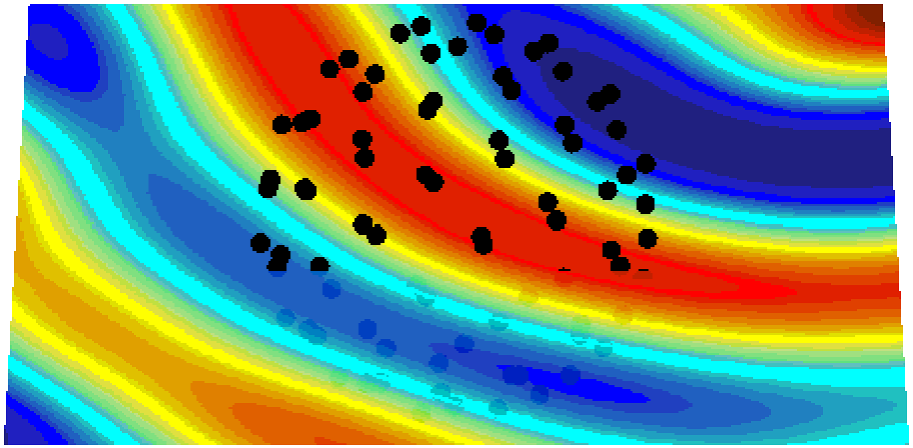

Array of Q microphones

- For each microphone *q*, located at r_q, θ_q, ϕ_q, it holds that

$$p_q(r,\theta,\phi) = \sum_{n=0}^{N} j_n(kr_q) \sum_{m=-n}^{n} Y_n^m(\theta_q,\phi_q) B_{mn} \quad \Rightarrow \quad \mathbf{p} = \tilde{\mathbf{Y}}\mathbf{J}\mathbf{b}$$

Matrix form

Cost function

$$J = \left\| \mathbf{p} - \tilde{\mathbf{Y}}\mathbf{J}\mathbf{b} \right\|^2 \quad \Rightarrow \quad \mathbf{b}_0 = \mathbf{J}^{-1}\tilde{\mathbf{Y}}^{+}\mathbf{p}$$

Estimated coefficients

Aliasing

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- The sound field is given by the sum of an infinite number of multipoles but a finite number of microphones is available

$$p(r,\theta,\phi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}B_{mn}\,j_n(kr)Y_n^m(\theta,\phi)$$

- A limited number of coefficients can be estimated
- If the measured sound field includes the contribution of high order multipoles, spatial aliasing might occur. This can be reduced if the distance between microphones is small in comparison to the wavelength.

Ill-conditioning of the recording problem

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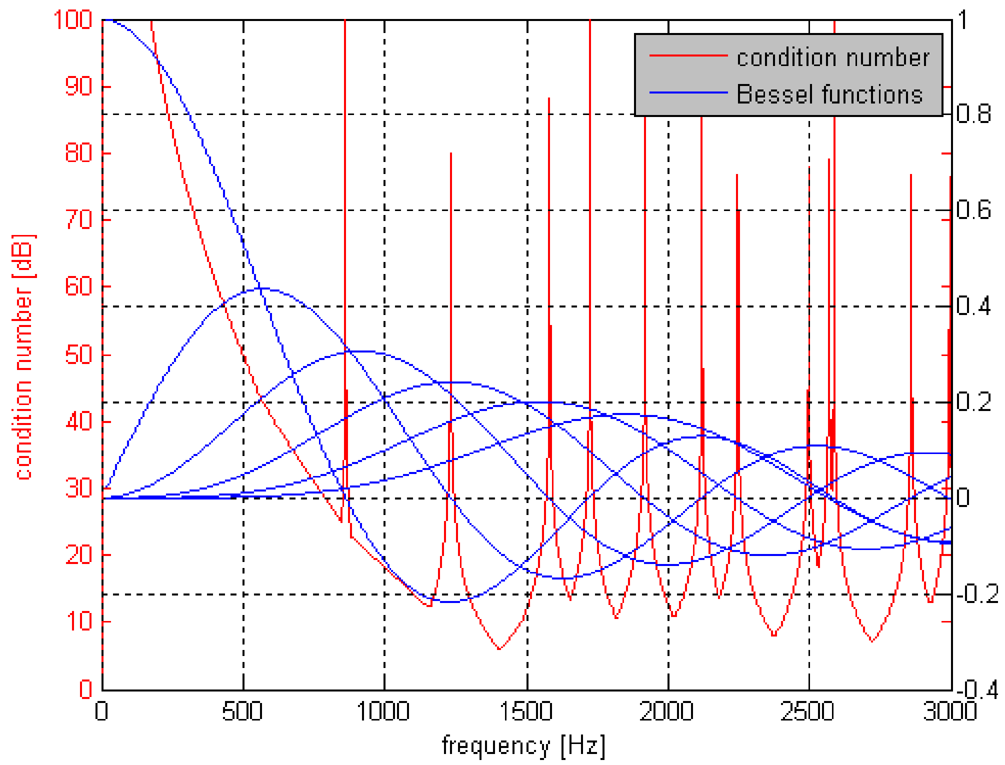
- Since the elements of the diagonal matrix **J** are spherical Bessel functions, its inverse can include elements which are very large or even infinity (especially if *kr* corresponds to the zero of one of the Bessel functions).
- Matrix **J** is likely to be ill-conditioned (rigid sphere arrays or arrays with directional microphones can be used to approach this problem)



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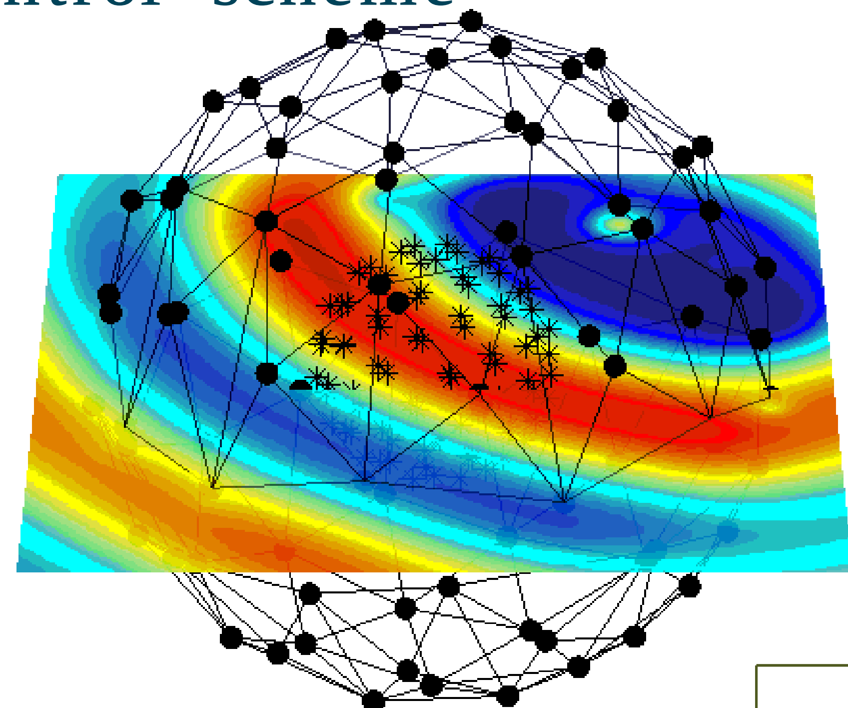
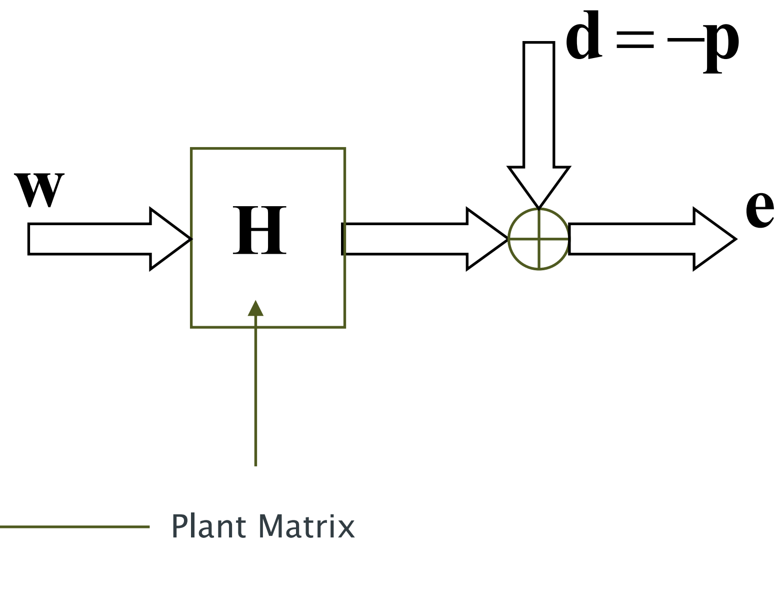
Connection to the Active Control of Sound

Sound reproduction using an “active control” scheme





Error function $$\mathbf{e} = \mathbf{H}\mathbf{w} - \mathbf{p}$$

Cost function $$J = \mathbf{e}^{\mathbf{H}}\mathbf{e} = \left\| \mathbf{p} - \mathbf{H}\mathbf{w} \right\|^2 \quad \Rightarrow \quad \mathbf{w}_0 = \mathbf{H}^{+}\mathbf{p}$$

“Array modes”

$$\mathbf{p} = \mathbf{H}\mathbf{w}$$

$$\mathbf{H} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathbf{H}}$$

$$(\mathbf{U}^{\mathbf{H}}\mathbf{U}) = \mathbf{I} \qquad (\mathbf{V}^{\mathbf{H}}\mathbf{V}) = \mathbf{I}$$

$$\mathbf{U}^{\mathbf{H}}\mathbf{p} = \boldsymbol{\Sigma}\mathbf{V}^{\mathbf{H}}\mathbf{w}$$

$$\tilde{\mathbf{p}} = \boldsymbol{\Sigma}\tilde{\mathbf{w}}$$

$$\kappa(\mathbf{H}) = \sigma_N / \sigma_1$$

Practical example

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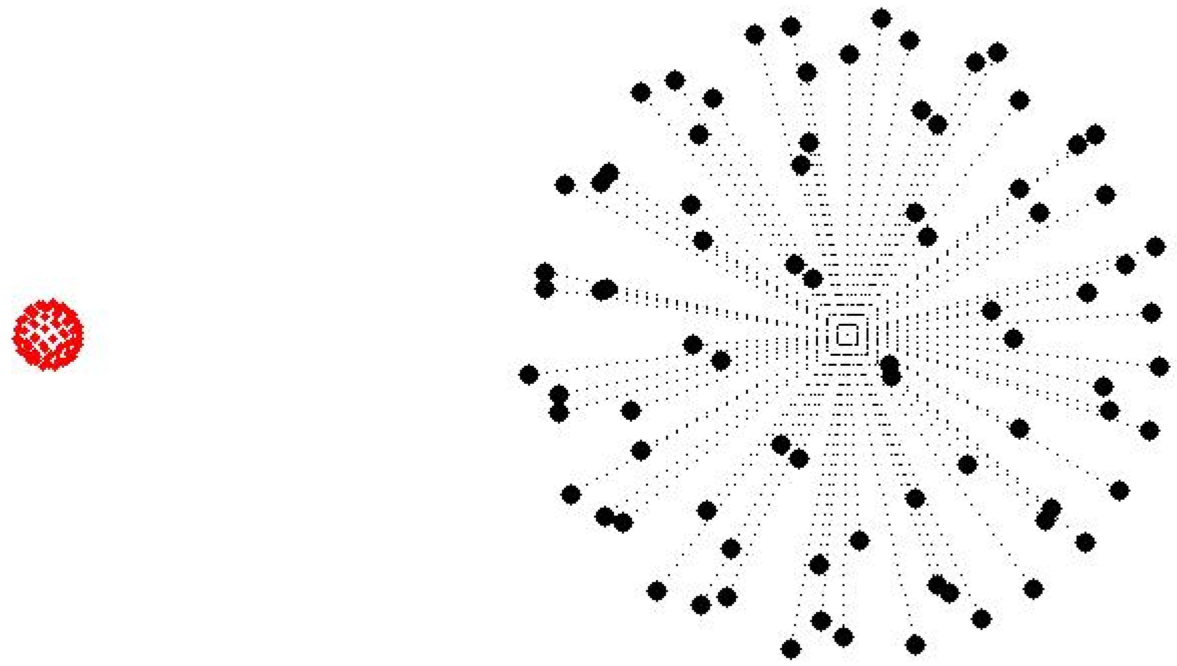
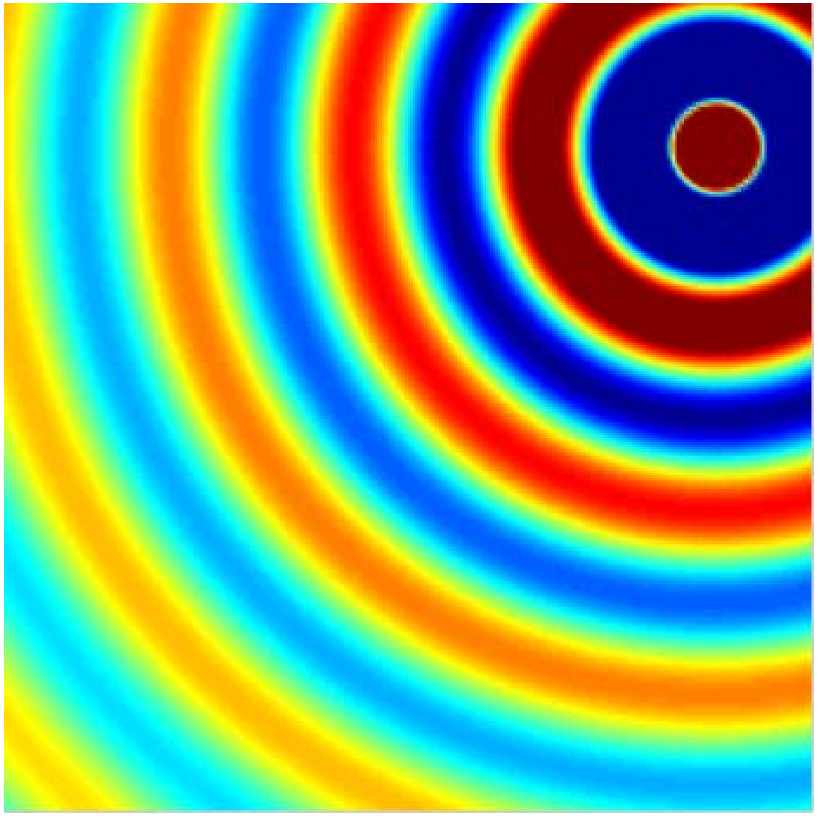
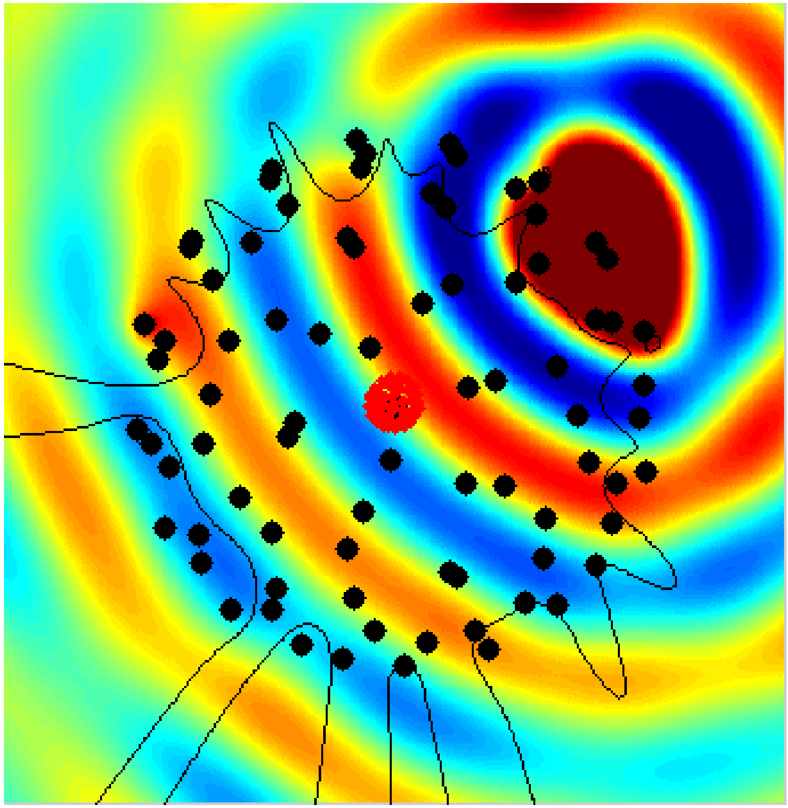


- Microphone array: 81 microphones, $r_2=0.2m$
- Loudspeaker array: 81 loudspeakers, $r_1=2m$



Original sound field

Reconstructed sound field



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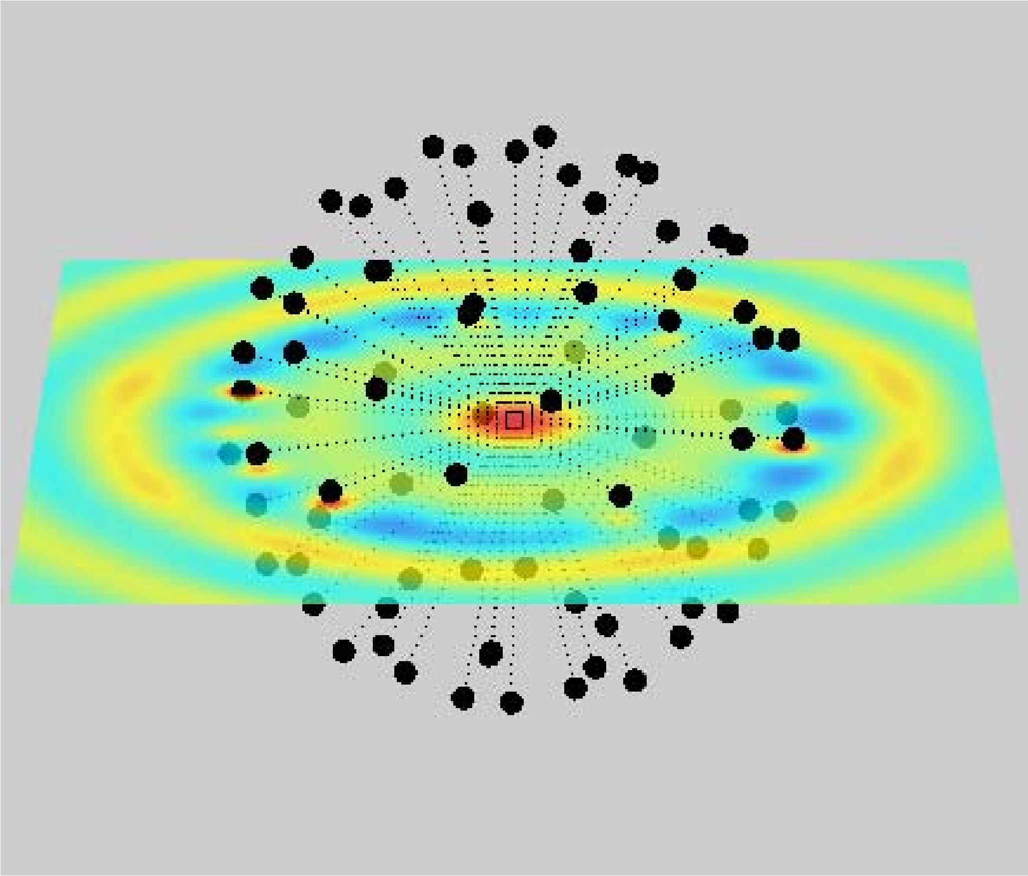
1st array mode (in phase)



σ_1=3.7458

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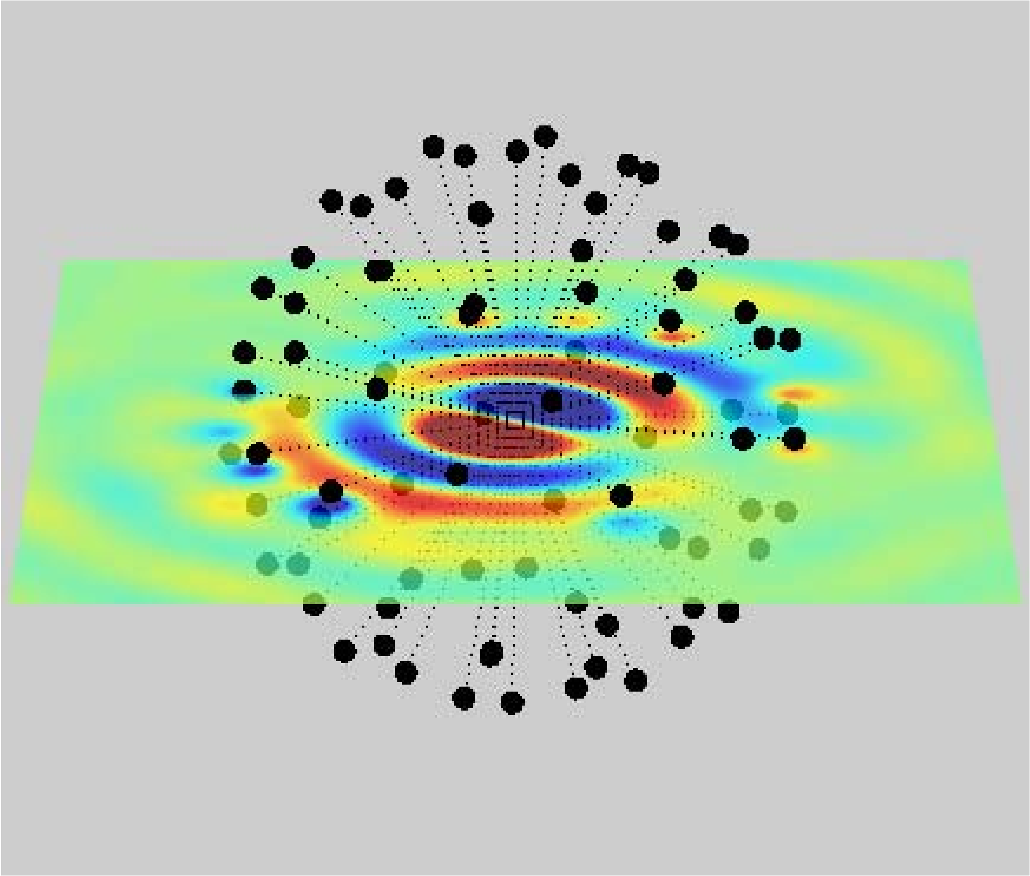
3rd array mode



σ_3=1.2528

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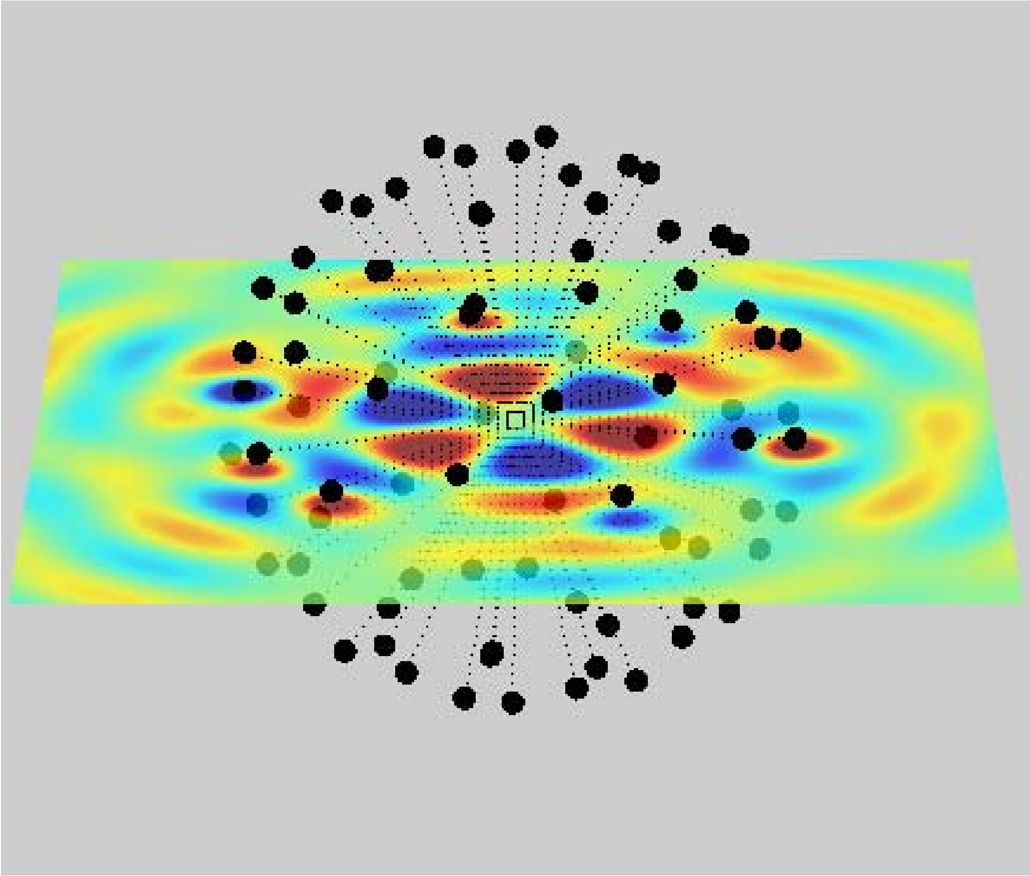
13^{th} array mode



σ_{13}=0.0153

Equivalence of the solutions

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- Use an SVD of the plant matrix:

$$\mathbf{H} = \mathbf{U}\Sigma\mathbf{V}^{\mathbf{H}}$$

- "Active control" solution

$$\mathbf{p} = \mathbf{H}\mathbf{w} \longrightarrow \mathbf{p} = \mathbf{U}\Sigma\mathbf{V}^{\mathbf{H}}\mathbf{w} \longrightarrow \mathbf{w} = \mathbf{V}\tilde{\Sigma}^{-1}\mathbf{U}^{\mathbf{H}}$$

- "Ambisonics" solution

$$\mathbf{p} = \tilde{\mathbf{Y}}\mathbf{J}\mathbf{b}$$

$$\mathbf{b} = \mathbf{R}\mathbf{Y}\mathbf{w}$$

$$\longrightarrow \mathbf{p} = \tilde{\mathbf{Y}}\mathbf{J}\mathbf{R}\mathbf{Y}\mathbf{w} \longrightarrow \mathbf{w} = \mathbf{Y}^{\mathbf{H}}\left(\frac{(4\pi)^2}{QL}\mathbf{J}^{-1}\mathbf{R}^{-1}\right)\tilde{\mathbf{Y}}^{\mathbf{H}}\mathbf{p}$$

If the matrices of spherical harmonics are unitary

Extension to continuous sources and sensors

Continuous source distribution

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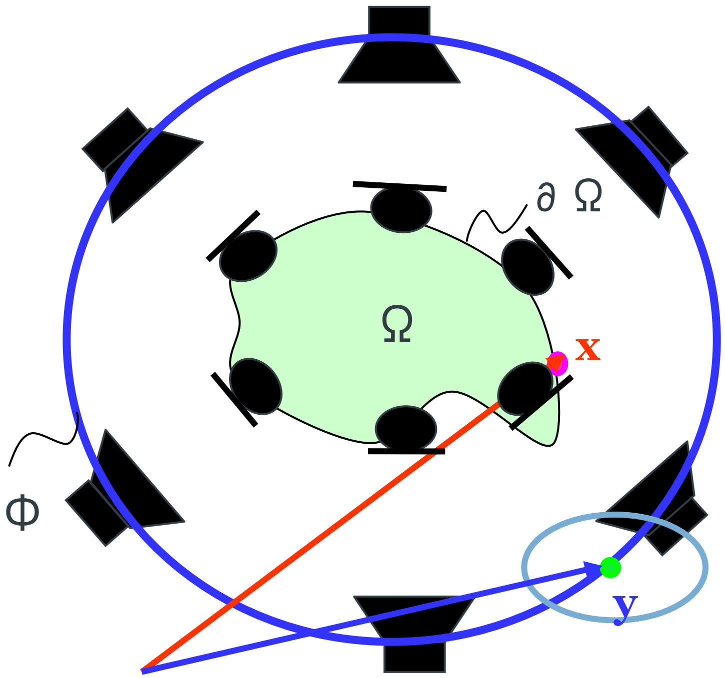

Acoustic pressure on the boundary ∂ Ω

Loudspeaker coefficients (aperture function)

$$p(\mathbf{x}) = \int_{\Phi} G(\mathbf{x}|\mathbf{y})\, w(\mathbf{y})\, dS(\mathbf{y})$$

(Free field) Green Function

- The loudspeaker and microphone arrays are represented by continuous distributions of monopole secondary sources on Φ and omnidirectional receivers on ∂Ω, respectively.
- The acoustic field $p(\mathbf{x})$ on ∂Ω uniquely determines the sound field within Ω for wave numbers that are not an eigenvalue of the negative Laplacian on Ω.
- The loudspeaker coefficients $w(\mathbf{y})$ are calculated as an approximate solution of an integral equation of the first kind.

The Integral Equation

$$p(\mathbf{x}) = \int_{\Phi} G(\mathbf{x}\,|\,\mathbf{y})\, w(\mathbf{y})\, dS(\mathbf{y})$$

- This integral equation represents an ill-posed problem
- The solution of the equation can be represented by using the singular value decomposition

$$w(\mathbf{y}) = \sum_{n=1}^{N} \frac{1}{\sigma_n} w_n(\mathbf{y}) \int_{\partial\Omega} p_n(\mathbf{x})^* \, p(\mathbf{x})\, dS(\mathbf{x})$$

(Fazi and Nelson, IOA 23rd conference on reproduced sound, 2007)

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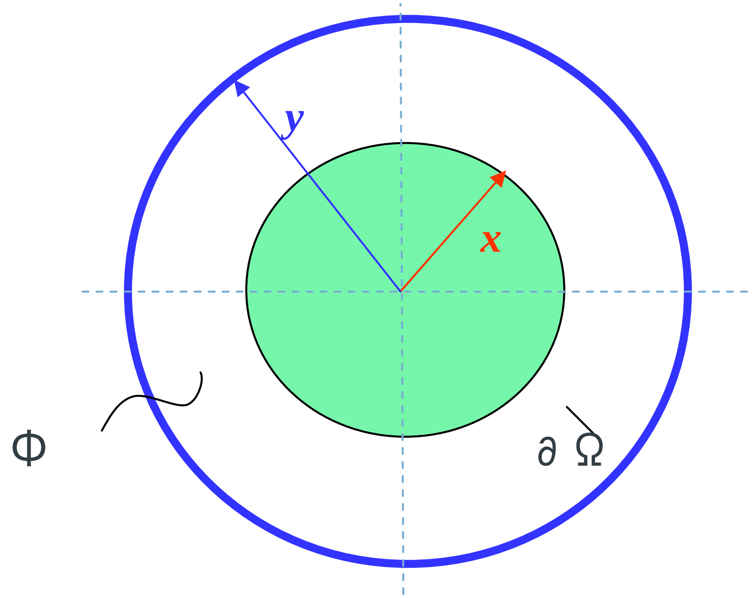
Concentric spheres



This equals 0 at the eigenvalues of the negative Laplacian on Ω

$$w(\mathbf{y}) = \sum_{n=0}^{N} \sum_{m=-n}^{n} Y_n^m(\hat{\mathbf{y}}) \frac{1}{(-jk)h_n^{(2)}\left(k|\mathbf{y}|\right) j_n\left(k|\mathbf{x}|\right)} \int_{\partial\Omega} Y_n^m(\hat{\mathbf{x}})^* p(\hat{\mathbf{x}}) dS(\hat{\mathbf{x}})$$

$$\mathbf{w} = \mathbf{V} \quad \boldsymbol{\Sigma}^{-1} \quad \mathbf{U}^{\mathbf{H}}\mathbf{p}$$

Conclusions

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- Given a sound field with a known spherical harmonic decomposition, the source strengths required for reproduction can readily be deduced (the “decoding” problem is well posed)
- Deducing the spherical harmonic decomposition of a sound field from measurement is more problematic (the “encoding” problem can be ill-posed)
- There are clear connections between “active sound control” and “ambisonics”
- The framework of linear algebra associated with discrete multi-channel systems extends naturally to the spatially continuous case

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Thank you